Thm If R, is a submfld of $S\times S$ and $\pi_1 : R \to S$ is a submersion,
is an equiv rel'n on S^d, $d-r$
Then the top space S/R has a smooth atlas of dim $d-r$

Corrected proof:

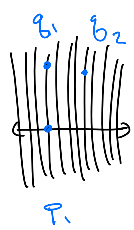
⌈ Suppose $p_0 \in S$. Let $N^{d-r} \subseteq S$ be s.t. $p\times N$ is a transversal to $R \cap p\times S$ (using π_1 a submersion).
Again by π_1 submersion, $R\cap N\times S$ is a manifold of dimension d, containing (p,p). Furthermore, one checks that $d\pi_2 : T_{(p,p)} R\cap N\times S \longrightarrow T_pS$ is an isomorphism, so π_2 is locally invertible. Then $\pi_1\circ\pi_2^{-1}$ and the inclusion i_N give inverse smooth maps, representing a chart for S at p ⌋

$p_0 \in S$

① $\exists\ \varphi : \mathcal{U} \subseteq \mathbb{R}^{d-r} \longrightarrow S$ s.t. $\varphi\times\varphi \pitchfork R$
$0 \longmapsto (p,p)$

② $R \underset{\text{codim } d-r}{\pitchfork} \varphi(\mathcal{U})^{d-r} \times S^d \Rightarrow R\cap\varphi(\mathcal{U})\times S$ is a d-manifold
$((p\sim q)$ s.t. $p\in\varphi(\mathcal{U}))$

pic of $\varphi(\mathcal{U})$ and $\{\pi_2\circ\pi_1^{-1}p\}_{p\in\mathcal{U}}$



③ $d\pi_2|_{(p,p)} : T_{(p,p)} R\cap\varphi(\mathcal{U})\times S \longrightarrow T_pS$ show injective

$(v,0) \in T_{(p,p)}R$

$\Rightarrow (0,v) \in T_{(p,p)}R$ (symmetry also instant — d(tr))

$\Rightarrow v \notin \ker(d\varphi)$

$\rightarrow\ \leftarrow$

$\Longrightarrow$ diffeo

(4) $\pi_1 \circ \pi_2^{-1} : S \to \psi(\mathcal{U})$ descends to S/R

$p \longmapsto p$

Pf

If $(q_1, q_2) \in R$, let

$(p_1, q_1) = \pi_2^{-1}(q)$

$(p_2, q_2) = \pi_2^{-1}(q')$

$\Rightarrow (p_1, p_2) \in R$ (transitivity)

Proof $\underset{\text{d-r}}{\psi(\mathcal{U})} \times \underset{\text{d-r}}{\psi(\mathcal{U})} \pitchfork \underset{\text{d-r}}{R} \Rightarrow \cap$ is a d-r-mfld

$\psi(\mathcal{U}) \times \psi(\mathcal{U}) \cap \Delta$ is a sub-d-r-mfld

$\Rightarrow$ open subset

$\Rightarrow$ after pass shrinking $\mathcal{U}$, i.e. it's exactly the $\cap$. $\lrcorner$

(5) $\pi_1 \circ \pi_2^{-1} : S/R \to \psi(\mathcal{U})$

and $\iota : \psi(\mathcal{U}) \to S \to S/R$ are inverse

Pf $\iota \circ (\pi_1 \circ \pi_2^{-1}) = \mathbb{1}_{S/R}$ ✓ trivial, unwind.

$(\pi_1 \circ \pi_2^{-1}) \circ \iota$ ✓ $\lrcorner$

Vector fields, Flows,

From last time:

- A vector field on $S^d \subseteq \mathbb{R}^n$ is $V : S \to S \times \mathbb{R}^n$ s.t. $V_p \in T_p S$
  $V(p) = (p, V_p)$
- A diffeo $\varphi : S \to \tilde{S}$ identifies $\mathrm{Vect}(S)$ with $\mathrm{Vect}(\tilde{S})$

$\Longrightarrow$ A chart identifies vector fields on S locally with vector fields on $\mathbb{R}^d$

Rmk This also works for quotients, or abstract mflds –

If ψ is a chart, x^i coords on $\mathbb{R}^d$, then
$\psi_*\left(\frac{\partial}{\partial x_i}\right)$ is a local vector field on S

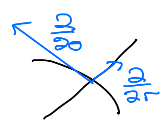
$$\psi_* \frac{\partial}{\partial x_i}(p) = \left(p, \frac{\partial \psi}{\partial x_i}(\psi^{-1}(p))\right)$$

Often omit $\psi \longrightarrow x^i$ local coords on S then $\frac{\partial}{\partial x^i}$ local vect field

Characterized by $\left(\frac{\partial}{\partial x^i}\right) x^j = \delta^j_i$

Warning: $\frac{\partial}{\partial x^i}$ depends on all coord fns

eg on $\mathbb{R}^2 - \{0\}$, coords $(r, \theta) \to$



Tangent bundle:

Dfn. The tangent bundle TS is the manifold

local charts from charts for S

$\{(p, v) \subseteq \mathbb{R}^{2n} \mid p \in S, v \in T_p S\}$

- It comes with a projection $\pi : TS \to S$
- A vector field is equivalent to a section of π $(\pi \circ V = \mathrm{id}_S)$

'graph of the vector fields'

not vec. arrows!

A vector bundle over S is a family of vector spaces smoothly varying over S, i.e. A submanifold $E \subseteq S \times \mathbb{R}^N$ some N s.t. each fiber of π_S is a r-dim'l subspace.

Rmk: Analogous abstract def'n, which we won't need.

Main example: pull-back bundle.

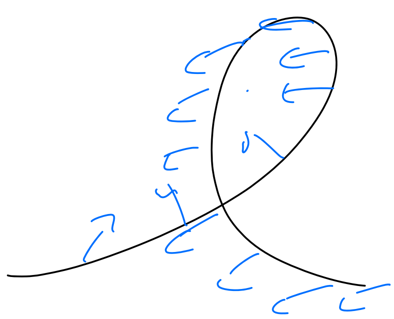
Defn

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "\pi"] \\ S \arrow[r, "f"] & \tilde{S} \end{tikzcd}$$

$\pi \pitchfork f \Rightarrow f^*TS$ is a mfld; fibers are vec. spaces.

b/c π already submersion

If $f : S \to \hat{S}$ is any smooth map, then $f^*T\hat{S}$ is a v.b. on S. What do sections look like?



Note if $E = S \times \mathbb{R}^n$ is trivial, then so is f^*E.

Lem Let e_i be a basis of sections of E. Then a general section of f^*E is $V = \sum_i V^i e_i \circ f$ where V^i fns on S.

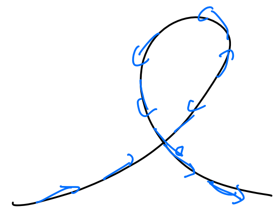
We can still define

$$\begin{tikzcd} TS \arrow[rr, "df"] \arrow[dr] & & f^*T\hat{S} \arrow[dl] \\ & S & \end{tikzcd}$$

(smooth) map of vector bundles

eg

$V = 1$ on $\mathbb{R}$



A section of $f^*T\hat{S}$ is an 'infinitesimal deformation of the map f.

1) Case of $S = pt$

2) Case of $S = \hat{S}$

In case (2), we can ask about flows.

FTODE

Let V be a v.f. on a mfld M, $q_0 \in M$. There is a nbhd $\mathcal{U}$ of q_0 and $\varepsilon > 0$ s.t. $\exists$ flow

$$\Phi : (-\varepsilon, \varepsilon) \times \mathcal{U} \longrightarrow M$$

solving ① $\Phi(q, 0) = q$

② $\dfrac{\partial \Phi}{\partial t} = V_{\Phi}$

$\quad = d\Phi(0 \times \frac{\partial}{\partial t})$

Φ is unique.

Restricting to q, you get the flow line $\Phi_q : (a, b) \longrightarrow M$ ($a, \mathbb{R}$)

Useful lemma: If the maximal existence time for flow is $T < \infty$, then as $t \to T$, $\Phi(q, t)$ leaves every cpt set.

Pf On a cpt ~~set~~ K, $\exists$ $\varepsilon(K)$ s.t. the flow exists to time ε $\forall q \in K$ hence taking $q = \Phi_q(T - \frac{\varepsilon}{2})$ gives a $\to\leftarrow$ by !

Cor Every vector field on a cpt mfld is complete.

gives $\mathbb{R}\times M \to M$.

<u>Rmk</u> ! Also implies $\Phi(t_1+t_2, p) = \Phi(t_1, \Phi(t_2, p))$ ($\Rightarrow \Phi$ is a diffeo!)

($\uparrow$ check that $\Phi(t_1+t_2, p)$ satisfies ① and ② for $q = \Phi(t_2, p)$)

$\rightsquigarrow$ the <u>group</u> $(\mathbb{R}, +)$ acts on M

Note that there is additional info in $\mathbb{R}\times M \xrightarrow{\text{smooth}} M$.